ON THE FORMULATION OF THE PROBLEM OF THE MAGNETOHYDRODYNAMIC BOUNDARY LAYER

(K POSTANOKE ZADACHI O MAGNITOGIDRODINAMICHESKOM POGRANICHNOM SLOE)

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At present there are quite a large number of papers devoted to various aspects of the problem of the magnetohydrodynamic boundary layer. The formulation of the problem of the magnetohydrodynamic boundary layer, for the case of small magnetic Reynolds numbers (R_m), can be found in [1], for the case of large R_m in [2-3]. In the course of development of these papers classes of solutions of the respective problems were investigated, and particular problems were solved. In the present paper, a systematic presentation is made of considerations and relationships concerning the formulation of the problem, (some of which are contained in one form or another in earlier papers). These considerations may be found useful in various complex cases arising in the formulation of the problem. In the interest of simplification, we shall consider only the boundary layer in an incompressible fluid.

Let us assume that, under the influence of either viscous or electromagnetic forces, or under the influence of both of these forces, a narrow region of flow develops in a stream, namely the boundary layer, in which the usual boundary layer assumptions for the velocity components are valid. Let δ be the thickness of that boundary layer, where $\delta/L \ll 1$, L being a characteristic length along the boundary layer.

If in the boundary layer under consideration viscous forces are appreciable, i.e. if the boundary layer is "viscous", then we obtain from the comparison of viscous and inertial terms in the equations of motion ($u\,\partial u/\partial x \sim \nu\,\partial^2 u/\partial y^2$) for the thickness of the boundary layer δ_ν

$$\delta \sim \delta_\nu \sim \frac{L}{\sqrt{R_L}}, \qquad R_L = \frac{UL}{\nu} \gg 1, \qquad R_\delta = \frac{U\delta}{\nu} \sim \frac{L}{\delta} \gg 1$$

Here and subsequently x and z denote the streamwise and normal coordinates within the boundary layer. Projections of vectors on the plane xz have the index τ, y is the coordinate across the layer. Projections of vectors on this axis in some instances are indexed n. If $R_\delta \lesssim 1$, then the layer will not be viscous ($\delta \ll \delta_\nu$). In the cases where the magnetic Reynolds number is

$$R_{mL} \gg 1 \quad (R_{mL} = 4\pi\sigma UL/c^2)$$

there may develop a "magnetic" boundary layer in the stream [2-3], i.e. a narrow layer, in which the magnetic field and the flow field of the electric current change sharply. If we define the thickness of the magnetic boundary layer, δ_j as the distance, over which sharp variations of the magnetic field and the field of currents across the boundary layer take place, then by equating terms of equal order of magnitude in the induction equation

$$\operatorname{rot}(\mathbf{v} \times \mathbf{H}) + \nu_m \Delta \mathbf{H} = (\mathbf{H} \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{H} + \nu_m \Delta \mathbf{H} = 0$$

we obtain

$$\frac{HU}{L} \sim \nu_m \frac{H}{\delta_j^2} \quad \text{for } R_{mL} \gg 1$$

or

$$\delta \sim \delta_j \sim \frac{L}{\sqrt{R_{mL}}} \quad \text{for } R_{m\delta} \equiv \frac{4\pi\sigma U\delta}{c^2} \sim \frac{L}{\delta} \gg 1$$

If $R_{m\delta} \gg L/\delta$, the flow inside the boundary layer is described by the equations of magnetohydrodynamics for $\sigma = \infty$. It is assumed that dissociation, as a consequence of electric currents, does not take place. Surface currents may also be encountered. The thickness of the boundary layer in this case is not related to the magnetic Reynolds number, but it is determined by some other factors, for example, by viscous forces.

The relative thickness of the viscous and the magnetic layers is determined for $R_{mL} \gg 1$ by the parameter [4-5]

$$\varepsilon \equiv \frac{\delta_\nu}{\delta_j} \sim \sqrt{\frac{R_{mL}}{R_L}} \sim \sqrt{\frac{\nu}{\nu_m}}$$

For $\varepsilon \sim 1$ the thicknesses of the viscous and the magnetic layers are comparable ($\delta \sim \delta_\nu \sim \delta_j$). This case is investigated in [2]. For $\varepsilon \ll 1$ ($\delta \sim \delta_j$, $\delta_\nu \ll \delta_j$) the boundary layer will be a magnetic layer in an ideal fluid (see, for example, [3]).

In this case there may exist in the boundary layer surfaces of velocity discontinuity corresponding to the viscous sublayer. Flow in a viscous boundary layer (sublayer) for $\varepsilon \ll 1$ depends upon the magnetic

Reynolds number, calculated from δ_v. For $\varepsilon \gg 1 (\delta \sim \delta_v,\ \delta_j \ll \delta_v)$ the layer under consideration will be viscous. In this case a magnetic sub-layer may turn out to be a surface current, while the flow in the boundary layer must be treated on the basis of the magnetohydrodynamic equations for $\sigma = \infty$. In all these cases ($R_{mL} \gg 1$) the widest portion of the stream, located outside the boundary layer, is described by the equations of magnetohydrodynamics for $\sigma = \infty$. Some approximate solutions of the boundary layer problems for $\varepsilon \gg 1$ and $\varepsilon \ll 1$ are given in [4-5].

If the parameters which define the problem are such that $R_{mL} \lesssim 1$, then $R_{m\delta} \ll 1$. If strong outside electric fields are also absent ($E \lesssim UH/c$), then by comparing the terms in Ohm's law

$$\frac{c}{4\pi}\,\mathrm{rot}\,\mathbf{H} = \sigma\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{H}\right)$$

the distance δ^* may be evaluated, in which the magnetic field changes appreciably. In the general case

$$\left|\frac{c}{4\pi}\,\mathrm{rot}\,\mathbf{H}\right| \lesssim \left|\sigma\frac{\mathbf{v}\times\mathbf{H}}{c}\right|$$

where the sign of the inequality corresponds to the case $R_{m\delta^*} > 1$. In the case of a small magnetic Reynolds number

$$\left|\frac{c}{4\pi}\,\mathrm{rot}\,\mathbf{H}\right| \sim \left|\sigma\frac{\mathbf{v}\times\mathbf{H}}{c}\right|$$

and the evaluation of the magnitude of δ^* yields

$$\frac{L}{\delta^*} \sim R_{mL} = R_{m\delta}\,\frac{L}{\delta}, \quad \text{for} \quad \frac{\delta}{\delta^*} \sim R_{m\delta} \ll 1$$

Hence it follows that the boundary layer under consideration will not be a current layer, because $\delta^* \gg \delta$. The thickness of the boundary layer in this case is determined by viscous forces and $\delta \sim L/\sqrt{R_L}$.

Note: If strong electrical fields are created by external agencies ($E \gg UH/c$), the hydrodynamic problem is simplified, because the electromagnetic force in this case is defined only by the "external" electric field and by the currents produced by it. Consequently, the electromagnetic force is a given quantity, since the distribution of the external fields and currents may be determined independently.

From the evaluations, given below (6) to (7), it will be seen that the convection currents ($\rho_e\mathbf{v}$) may be neglected in Ohm's law. We shall merely note, that these currents may not lead to an appreciable change of the magnetic field in the boundary layer, since, when comparing the terms $c/4\pi$ rot $\mathbf{H}$ and $\rho_e\mathbf{v}$ (expression for ρ_e see below (1)), we find that $\delta/\delta^* \sim U^2/c^2$.

Hence, for $R_{mL} \lesssim 1$ the currents which flow in the boundary layer are small, so that they will not lead to an appreciable change of the magnetic field. The magnitude of the magnetic field is determined from the solution of the problem outside of the boundary layer. (If the boundary layer is assumed to occur on a body, then the free stream problem comprises the solutions of the problems of flow of an ideal fluid past an immersed body and the problem of finding the field inside the body.) If $R_{mL} \ll 1$, then the magnetic field in the entire stream may be considered to be a given quantity (in the problem of flow past an immersed body it is determined by the currents flowing in the immersed body). The formulation of such a problem, in the particular case where a magnetic field is given, is presented in [1]. If $R_{mL} \sim 1$, then the equations of magnetohydrodynamics must be solved in the stream outside of the boundary layer and the solution for the magnetic field must be matched (in the case of flow past an immersed body) with the solution for the distribution of the magnetic field in the body.

The influence of the electromagnetic field on the flow in a boundary layer in the case $R_{mL} \lesssim 1$ is signified by the presence of the electromagnetic force

$$\mathbf{f} = \rho_e \mathbf{E} + \frac{1}{c}\,\mathbf{j} \times \mathbf{H} = \rho_e \mathbf{E} + \frac{\sigma}{c}\left(\mathbf{E} + \frac{\mathbf{v} \times \mathbf{H}}{c}\right) \times \mathbf{H}$$

where in the case of flow in a boundary layer $\mathbf{H}$ is a given quantity which depends on the coordinates within the boundary layer. Consequently, in order to solve the problem of a boundary layer for $R_{mL} \lesssim 1$, it is necessary either to know the distribution of the volume charge ρ_e and of the electric field $\mathbf{E}$ inside the boundary layer, or else to include equations by which these quantities may be determined. By applying the operation div to Ohm's law, we obtain the distribution of the volume charge

$$4\pi\rho_e = \operatorname{div} \mathbf{E} = -\frac{1}{c} \operatorname{div} (\mathbf{v} \times \mathbf{H}) = -\frac{1}{c} (\mathbf{H} \operatorname{rot} \mathbf{v} - \mathbf{v} \operatorname{rot} \mathbf{H}) \qquad (1)$$

The electric field is described by the Maxwell equations

$$\operatorname{rot} \mathbf{E} = 0, \qquad \operatorname{div} \mathbf{E} = 4\pi\rho_e \qquad (2)$$

and it depends in general upon the distribution of the charge density in the boundary layer, in the free-stream and, in the case of flow about a body, in the body.

In the case investigated in [1] of plane flow, with $\mathbf{H}$ = const and directed in the plane of flow and $\rho_e \equiv 0$ in the free-stream as well as in the boundary layer. Also $\mathbf{E} \equiv 0$ in the whole stream and the electromagnetic force is defined only by the velocity field and the magnetic field.

If $R_{mL} \ll 1$, then the magnetic field is determined by the currents which flow outside the region of flow and, consequently, in the boundary layer and in the free stream rot $\mathbf{H} = 0$. It is to be noted that condition $\mathbf{H} = 0$ in this case does not imply that the current density is zero.

If $R_{mL} \sim 1$, then rot $\mathbf{H} \neq 0$ and in the boundary layer

$$\mathrm{rot}\,\mathbf{H} = \frac{4\pi\sigma}{c}\left[\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{H}\right]$$

By evaluation in the boundary layer and by using the fact that in the boundary layer $\partial\mathbf{H}/\partial y = 0$, we obtain for the charge density:

$$-4\pi c\rho_e = \mathbf{H}\cdot\mathrm{rot}\,\mathbf{v} - \mathbf{v}\cdot\mathrm{rot}\,\mathbf{H} = \mathbf{H}_\tau\,\mathrm{rot}_\tau\,\mathbf{v} + \mathbf{H}_n\cdot\mathrm{rot}_n\,\mathbf{v} - \frac{4\pi\sigma}{c}\mathbf{v}\cdot\mathbf{E}\varkappa = \tag{3}$$

$$= \frac{\partial}{\partial y}|\mathbf{v}_\tau\times\mathbf{H}_\tau| + H_n\,\mathrm{rot}_n\,\mathbf{v} - \frac{4\pi\sigma}{c}(\mathbf{v}_\tau\cdot\mathbf{E}_\tau)\varkappa + O(\delta), \qquad \varkappa = \begin{cases}1 & \text{for } R_{mL}\sim 1\\ 0 & \text{for } R_{mL}\ll 1\end{cases}$$

Hence it follows that the charge density in the boundary layer

$$\rho_e \sim \frac{HU}{c}\frac{1}{\delta} \sim \frac{1}{\delta} \quad \text{for} \quad \frac{\partial}{\partial y}|\mathbf{v}_\tau\times\mathbf{H}_\tau| \neq 0 \tag{4}$$

$$\rho_e \sim \frac{1}{c}\frac{UH}{L} \lesssim 1 \quad \text{for} \quad \frac{\partial}{\partial y}|\mathbf{v}_\tau\times\mathbf{H}_\tau| = 0 \tag{5}$$

If we consider the boundary layer on the immersed body, then on the body $|\mathbf{v}_\tau \times \mathbf{H}_\tau| = 0$, because $\mathbf{v}_\tau = 0$. Consequently, Equation (4) corresponds to the case $|\mathbf{v}_\tau \times \mathbf{H}_\tau|_\infty \neq 0$, and (5) corresponds to the case $|\mathbf{v}_\tau \times \mathbf{H}_\tau|_\infty = 0$.

We shall show that in spite of the presence of the charge density inside the boundary layer, we may neglect the "electric" force $\rho_e\mathbf{E}$ in all cases of practical importance. Indeed when comparing terms $\rho_e\mathbf{E}$ and $(\sigma/c)\mathbf{E}\times\mathbf{H}$ in the force expression, we obtain

$$|\rho_e\mathbf{E}| \ll \left|\frac{\sigma}{c}\mathbf{E}\times\mathbf{H}\right| \quad \text{for} \quad \begin{cases}\dfrac{U}{\sigma L} \ll 1, \quad \text{if} \quad \dfrac{\partial}{\partial y}|\mathbf{v}_\tau\times\mathbf{H}_\tau| = 0 & (6)\\[2ex] \dfrac{U}{\sigma\delta} \sim \dfrac{U^2}{c^2}\dfrac{\sqrt{R_L}}{R_{mL}} \ll 1, \quad \text{if} \quad \dfrac{\partial}{\partial y}|\mathbf{v}_\tau\times\mathbf{H}_\tau| \neq 0 & (7)\end{cases}$$

Inequality (6) is always considered to be valid, because it is predicated upon the neglect of the displacement currents [6]. Inequality (7) may be violated for $R_{mL} \lesssim 1$, if the viscosity is very small $\delta \sim U^2L/c^2$. But in this case the boundary layer does not really exist since it is transformed into a charged vorticity layer in an ideal fluid. If a viscous boundary layer exists, inequality (7) may be violated only in the case of very small $R_{mL}(R_{mL} \sim U^2\sqrt{(R_L/c^2)})$. In this case, by comparing the magnetic with viscous forces in the equations of motion, it is found that the electromagnetic forces will exert an influence upon the flow in the viscous boundary layer, if

$$\nu \frac{U}{\delta^2} \sim \frac{\sigma}{\rho c^2} UH, \quad \text{or} \quad R_{mL}\frac{H^2}{\rho U^2} \sim \frac{H^2}{\rho c^2}\sqrt{R_L} \sim 1 \tag{8}$$

i.e. in the case of exceedingly large magnetic fields. Let it be noted that Equation (8) may be obtained by relating the electric force $\rho_e \mathbf{E}$ to the viscous forces, if the evaluations (4), (5) and the relation $E \lesssim UH/c$ are used.

In this manner, the electromagnetic force for the flow in a boundary layer is

$$\mathbf{f} = \frac{\sigma}{c}\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{H}\right) \times \mathbf{H} \tag{9}$$

When comparing the electromagnetic force (9) with the inertia forces, it is found that the electromagnetic forces exert an influence upon the flow in the boundary layer for $R_{mL} \lesssim 1$, if [1]

$$mL = \frac{\sigma H^2 L}{c^2 U \rho} \sim R_m \frac{H^2}{\rho U^2} \sim 1$$

Instead of this parameter a parameter may be used which arises from the ratio of the viscous and the electromagnetic forces

$$\frac{{M_L}^2}{R_L} \sim 1, \qquad M = \frac{HL}{\sqrt{4\pi\rho\nu\nu_m}} \quad (M \text{ is the Hartman number}) \tag{10}$$

Relationship (10) shows that in the case of small Hartman numbers ($M \ll \sqrt{R_L}$) the electromagnetic forces do not influence the flow in the boundary layer.

Hence, to solve the problem of the boundary layer an additional force must be introduced into the equation of the boundary layer, which is determined by Equation (9) and in addition also by Equation (2). But since the electric field in the boundary layer depends upon the external charge distribution, and the boundary conditions for the solution of the external problem depend upon the charge concentrated in the boundary layer, the equations of the boundary layer and the free stream in general may not be solved independently. On the basis of certain considerations and the investigations developed below, it is possible, however, first to simplify Equation (2) within the limits of the system of boundary layer equations and, second, to formulate the boundary conditions for the external problem so that it would be possible to solve the external problem and the boundary layer equations separately.

In setting up the equations of motion in the boundary layer it is necessary to take into account only the main (in terms of δ) terms in the Expression (9). Therefore, if any quantity in (9) is of order of magnitude δ in the boundary layer, then the terms which contain this quantity may be neglected. On the other hand, if any quantity undergoes a variation of the order of δ inside the boundary layer, then the value of this quantity must appear in (9) as obtained from the solution of the external problem while disregarding the boundary layer.

The quantity $\mathbf{H}$ in (9) is calculated from the solution of the external problem. For $R_{mL} \lesssim 1$ it does not depend on the flow in the boundary layer, therefore $\mathbf{H}$ must be assumed to be of the order of 1 (with respect to δ) for the flow in the boundary layer. Since in the boundary layer $u \sim w \sim 1$ and $v \sim \delta$, the expression $u\mathbf{e}_x + w\mathbf{e}_z$ is to be substituted for the velocity in (9).

Let the principal part of the electric field which is used for calculating the forces in the boundary by Equation (9) be denoted by $\mathbf{E}^{\circ}$. The expression for the force in the boundary layer equations in this case has the form

$$\mathbf{f} = \frac{\sigma}{c}(\mathbf{E}^{\circ} + u\mathbf{e}_x + w\mathbf{e}_z) \times \mathbf{H} \tag{11}$$

The electric field in the boundary layer is determined by the solution of the external problem and the distribution of charge inside the boundary layer. If $\partial|\mathbf{v}_T \times \mathbf{H}_T|/\partial y = 0$, then because of (5) the order of $\rho_e \sim 1$ and the summed charge of the boundary layer will be a quantity of the order of δ. Evidently, the variation of the electric field produced by the external sources on account of the charge concentrated inside the boundary layer, will be of the order of δ. Consequently, for $\partial|\mathbf{v}_T \times \mathbf{H}_T|/\partial y = 0$ the electric field $\mathbf{E}^{\circ}$ in (11) must be regarded to be the field produced externally in relation to factors residing in the boundary layer while in solving the external problem $\mathbf{E}^{\circ}$ is to be taken as a continuous function at the location of the boundary layer. Therefore, if $\partial|\mathbf{v}_T \times \mathbf{H}_T|/\partial y = 0$, then in the solution of the boundary layer problem $\mathbf{E}^{\circ}$ in Expression (11) will be a known function which is dependent only on the coordinate along the boundary layer and is determined by the solution of the external problem.

In the case of the boundary layer along a body the matter under consideration corresponds to the condition $|\mathbf{v}_T \times \mathbf{H}_T|_{\infty} = 0$. In this case if the immersed body is a dielectric, $j_n = E_n^{\circ} = 0$, must be taken as the boundary condition on the body for the external problem. But if the body is a conductor the condition is $\mathbf{E}_T^{\circ} = 0$.

If $\partial|\mathbf{v}_T \times \mathbf{H}_T|/\partial y \neq 0$ then, as a consequence of (4), $\rho_e \sim \delta^{-1}$ and the total charge of the boundary layer is of the order 1. Also, the charge concentrated in the boundary layer influences materially the external electric field and the variation of the electric field inside the boundary layer may be a quantity of the order of 1. Consequently, $\mathbf{E}^\circ$ in (11) will be a function of the coordinates inside the boundary layer, and to render this problem complete in this case, Equation (2) is to be added to the boundary layer equations. Using the values pertaining to the boundary layer, the system (2) is simplified by retaining only the main terms ($\mathbf{E}^\circ$) in it.

Since the boundary layer is a narrow charged layer, the component of the gradient of the electric field ΔE_n normal to the layer in this case is of the order

$$\Delta E_n \sim 4\pi \int_0^\infty \rho_e \, dy \sim 4\pi \frac{UH}{c}$$

The tangent component of the electric field is

$$E_\tau \lesssim \frac{1}{c} UH$$

Consequently, in the calculation of the principal part of the electric field $\mathbf{E}^\circ$, instead of the second Equation (2), we may use the equation

$$\frac{\partial E_n^\circ}{\partial y} = 4\pi \rho_e^\circ = -\frac{1}{c} \frac{\partial \,|\mathbf{v}_\tau \times \mathbf{H}_\tau|}{\partial y} \tag{12}$$

In analogous evaluations of the equation rot $\mathbf{E}^\circ = 0$, it is found that the variation of the tangent component of the electric field $|\Delta \mathbf{E}_\tau|$ across the boundary layer is of the order of δ

$$\left|\Delta \mathbf{E}_\tau\right| \sim \frac{\partial E_n}{\partial s} \delta$$

Consequently, in making calculations of the force (11) instead of the first Equation (2) the following equation may be used

$$\partial \mathbf{E}_\tau^\circ \,/\, \partial y = 0 \tag{13}$$

Equations (12), and (13) have a simple physical meaning, namely of the continuity of the tangent component and of the variation of the normal component of the electric field proportional to the charge density at the transition through the narrow charged layer. By integration of (12), we obtain

$$E_n^\circ = -\frac{1}{c} |\mathbf{v}_\tau \times \mathbf{H}_\tau| + f(x, z) \tag{14}$$

and, using Ohm's law

$$j_y^\circ = \sigma \left(E_y^\circ + \frac{1}{c} \;|\mathbf{v}_\tau \times \mathbf{H}_\tau|\right) = \sigma f(x, z)$$

The last relationship shows that in the boundary layer equation $\partial j_y^{\circ}/\partial y = 0$ is valid. This permits the boundary conditions for the external problem to be formulated not in terms of the electric field, but in terms of the charge density. Hence it follows, that for the determination of $\mathbf{E}^{\circ}$ in this case $(\partial|\mathbf{v}_\tau \times \mathbf{H}_\tau|/\partial y \neq 0)$ the external problem must be solved with the assumption that at the points which correspond to the boundary layer there is a layer, the surface charge of which has the density

$$q = -\frac{1}{4\pi c}(|\mathbf{v}_\tau \times \mathbf{H}_\tau|_{y=+0} - |\mathbf{v}_\tau \times \mathbf{H}_\tau|_{y=-0})$$

In the case of the boundary layer on a body, this corresponds to the condition $|\mathbf{v}_\tau \times \mathbf{H}_\tau|_\infty \neq 0$. If the immersed body is dielectric, then the condition $j_y^{\circ} = \sigma f(x, z) = 0$ serves as a boundary condition on the body for the solution of the external problem. If the immersed body is a conductor, then the outside problem must be solved using the condition $\mathbf{E}_\tau^{\circ} = 0$ on the surface of the body while the function $f(x, z) = \sigma^{-1} j_y$ for $y = 0$ is defined in this case by the solution of this problem.

If $\partial|\mathbf{v}_\tau \times \mathbf{H}_\tau|/\partial_y = 0$, then Equations (12) and (13) express the fact that $\mathbf{E}^{\circ}$ is not changed in the boundary layer, that is, in order to solve for $\mathbf{E}^{\circ}$ in Equation (11) the external problem must be solved in this case as already shown above, assuming that the charge concentrated in the boundary layer is zero. In [8] which was published very recently, analogous derivations were obtained for the boundary layer problem in a medium with anisotropic conductivity.

Let it be noted once more that the given considerations refer not to the calculation of the distribution of the electric field in the boundary layer, but only to the calculation of its principal part, which is required for the determination of the force in the boundary layer equations.

Projection of the momentum equation on the normal to the boundary layer y yields

$$-\frac{\partial p}{\partial y} + \frac{1}{c}|\mathbf{j}_\tau \times \mathbf{H}_\tau| = 0$$

i.e. the variation of the pressure (Δp) across the boundary layer is

$$\Delta p \lesssim \delta$$

and we may use equation $\partial p/\partial y = 0$ when solving the problem of the boundary layer.

Summing up the considerations above we write the system of equations which describe the flow (velocity field) in the boundary layer for the case $R_{mL} \leqslant 1$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-\nu\frac{\partial^2 u}{\partial y^2}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\sigma}{c\rho}\left(\mathbf{E}^\circ+\frac{1}{c}\mathbf{v}_\tau\times\mathbf{H}\right)\times\mathbf{H}\,|_x$$

$$\frac{\partial p}{\partial y}=0$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}-\nu\frac{\partial^2 w}{\partial y^2}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{\sigma}{c\rho}(\mathbf{E}^\circ+\frac{1}{c}\mathbf{v}_\tau\times\mathbf{H})\times\mathbf{H}\,|_z \quad (15)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

$$E_n{}^\circ=-\frac{1}{c}\mathbf{v}_\tau\times\mathbf{H}_\tau+f(x,z),\qquad \mathbf{E}_\tau{}^\circ=\mathbf{E}_\tau{}^\circ(x,z),\qquad \mathbf{H}=\mathbf{H}(x,z)$$

Functions $f(x, z)$, $\mathbf{E}_\tau{}^\circ(x, z)$, $\mathbf{H}(x, z)$ are determined from the boundary conditions and from the solution of the external problem.

In the case $R_{mL} \ll 1$ the external problem for the boundary layer on a body reduces to the equations

$$\rho\frac{d\mathbf{v}}{dt}=-\operatorname{grad}p+\frac{\sigma}{c}\left(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{H}\right)\times\mathbf{H},\qquad \operatorname{div}\mathbf{v}=0$$

$$\operatorname{rot}\mathbf{E}=0,\qquad \operatorname{div}\mathbf{E}=-\frac{1}{c}\operatorname{div}(\mathbf{v}\times\mathbf{H}) \quad (16)$$

with the boundary conditions for $y = 0$

$v_y=v_n=0,$	$E_y=-\frac{1}{c}\,\lvert\mathbf{v}_\tau\times\mathbf{H}_\tau\rvert_{y=0}$	(for flow past an immersed dielectric)
$v_n=0,$	$\mathbf{E}_\tau\,\lvert_{y=0}=0$	(for flow past an immersed conductor)

$\mathbf{H}$ in the given system is determined from the independent system

$$\operatorname{rot}\mathbf{H}=\frac{4\pi}{c}\mathbf{j}^*,\qquad \operatorname{div}\mathbf{H}=0 \quad (17)$$

where $\mathbf{j}^*$ is the current density outside the region of flow.

The distribution of an electric field in the body is found from the equations

$$\operatorname{rot}\mathbf{E}=0,\qquad \operatorname{div}\mathbf{E}=4\pi\rho_e^*$$

where $\rho_e{}^*$ is the charge density in the body (in a conductor $\rho_e = 0$). The boundary condition for the solution of this system is represented by the value of $\mathbf{E}$ on the body obtained from the solution of the system (16). The quantity derived from that equation defines the surface density of electric charge E_n which forms on the surface of the immersed body, because in the solution of the boundary layer problem on a dielectric $E_n^{\circ}|_{y=0} = E_n^{\circ}|_{y=0} = 0$, and on a conductor $E_n^{\circ}|_{y=0} = f(x, z)$, where $f(x, z)$ is defined by the solution of system (16).

In the case $R_{mL} \sim 1$ the external boundary layer problem on the body is reduced to the equations:

in the region of flow

$$\rho \frac{d\mathbf{v}}{dt} = -\operatorname{grad} p + \frac{1}{4\pi}(\operatorname{rot} \mathbf{H} \times \mathbf{H}), \qquad \operatorname{div} \mathbf{v} = 0$$
$$\operatorname{rot}(\mathbf{v} \times \mathbf{H}) + \nu_m \Delta \mathbf{H} = 0, \qquad \mathbf{E} = \frac{1}{c}(\nu_m \operatorname{rot} \mathbf{H} - \mathbf{v} \times \mathbf{H}) \tag{18}$$

outside the region of flow

$$\operatorname{rot} \mathbf{H} = \frac{4\pi}{c}\mathbf{j}^*, \qquad \operatorname{div} \mathbf{H} = 0, \qquad \operatorname{rot} \mathbf{E} = 0, \qquad \operatorname{div} \mathbf{E} = 4\pi\rho_e^* \tag{19}$$

If the immersed body is a dielectric, then

$$v_n = 0, \qquad \operatorname{rot}_n \mathbf{H} = 0 \quad \text{for } y = 0$$

and the solutions of the systems (18) and (19) must be harmonized for $y = 0$ (at the points that correspond to the boundary layer) with the conditions of continuity of the functions $\mathbf{E}_\tau$ and $\mathbf{H}$.

If the immersed body is a conductor, then

$$v_n = 0, \qquad \mathbf{E}_\tau = 0 \quad \text{for } y = 0$$

and the solutions of systems (18) and (19) must be harmonized for $y = 0$ by the conditions of continuity of the functions $\mathbf{E}_\tau$, $\mathbf{H}$, rot $\mathbf{H}$.

Some of the partial solutions of the boundary layer problem on an electrode are contained in [7]. By the use of certain additional assumptions these solutions may be considered to represent the solutions of the system (15) and (16) to (17).

The solution of the system (15) defines the velocity field in the boundary layer. In order to determine the current field in the boundary layer from Ohm's law

$$\mathbf{j} = \sigma\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{H}\right) \tag{20}$$

besides the solution of (15) the distribution of $\mathbf{E}$ in the boundary layer must be known.

In first approximation, as shown above, the current density may be defined by the equation

$$\mathbf{j} = \sigma\left[\mathbf{E}^\circ + \frac{H}{c}(u\mathbf{e}_x + w\mathbf{e}_z)\right]$$

where $\mathbf{E}^\circ$ is determined from Equations (12) and (13). If more detailed information on current distribution in the boundary layer is desired, terms of the order in Equation (20) must be taken into account. Hence, to find the velocity, the complete solution of the system (15) is required by taking into account the normal component of the velocity. For the determination of $\mathbf{E}$ the system of equations is to be utilized which permits calculating $\mathbf{E}$ to terms of the order of δ.

Such a system may be obtained from (2) and (3) by making corresponding evaluations. Let us represent $\mathbf{E}$ in the form of a sum $\mathbf{E} = \mathbf{E}^\circ + \mathbf{E}^1$. From the foregoing considerations it follows that $\mathbf{E}^1 \leqslant \delta$. Substituting $\mathbf{E} = \mathbf{E}^\circ + \mathbf{E}^1$ in the system (2) and evaluating the order of the derivatives, we obtain for $\mathbf{E}^\circ$ the system (12) and (13) and for $\mathbf{E}^1$ the following equations:

$$\frac{\partial E_n^1}{\partial y} = -\frac{1}{c} H_n \operatorname{rot}_n \mathbf{v} + \frac{4\pi\sigma}{c^2} \mathbf{v}_\tau \mathbf{E}_\tau^\circ \varkappa - \frac{\partial E_x^\circ}{\partial x} - \frac{\partial E_z^\circ}{\partial z}$$

$$\frac{\partial \mathbf{E}_\tau^1}{\partial y} = 0 \qquad \text{for } \frac{\partial |\mathbf{v}_\tau \times \mathbf{H}_\tau|}{\partial y} = 0 \tag{21}$$

$$\frac{\partial E_n^1}{\partial y} = -\frac{1}{c} H_n \operatorname{rot}_n \mathbf{v} + \frac{4\pi\sigma}{c^2} \mathbf{v}_\tau \mathbf{E}_\tau^\circ \varkappa - \frac{\partial E_x^\circ}{\partial x} - \frac{\partial E_z^\circ}{\partial z}$$

$$\frac{\partial E_x^1}{\partial y} = \frac{\partial E_y^\circ}{\partial x}, \qquad \frac{\partial E_z^1}{\partial y} = \frac{\partial E_y^\circ}{\partial z} \qquad \text{for } \frac{\partial}{\partial y} |\mathbf{v}_\tau \times \mathbf{H}_\tau| \neq 0 \tag{22}$$

Arbitrary functions arising from the integration of (21) or (22) must be determined from the solution of the external problem of the electric field; also the charge density in the external region must vanish (the external charge having been accounted for in the determination of $\mathbf{E}^\circ$).

Let us illustrate the above by a very simple example which has a trivial solution. Consider the flow of an ideal fluid past a flat dielectric plate. Let $R_{mL} \ll 1$ and let the magnetic field, which is determined only by external sources, be homogeneous and directed along the axis z – $\mathbf{H} = H_0\mathbf{e}_z$. External electric fields are absent. It is easy to prove that the following solution

$$u = U, \quad v = 0, \quad w = 0, \quad \mathbf{j} = 0, \quad E_y = c^{-1}UH_0 \tag{23}$$

$$E_x = 0, \quad E_z = 0, \quad \rho_e = 0, \quad p = \text{const in the stream } \mathbf{E} \equiv 0 \text{ in the body}$$

satisfies the system (16). This solution describes a translational flow, in which electric currents are absent. A surface charge is formed on the plate of the density $(1/4\pi c)UH_0$, which establishes an electric field in the external stream, balancing the inductive field.

We shall look for the solution of the boundary layer problem such that $w = 0$, $\partial/\partial z = 0$, $E_z = 0$. In this case the exact equations (1) and (2) have the form

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = -\frac{1}{c} H_0\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right), \qquad \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} = 0$$

and their solution, with consideration of the equation div $\mathbf{v} = 0$, has the form

$$E_x = -\frac{H_0}{c} v, \qquad E_y = \frac{H_0}{c} u \tag{24}$$

where $\mathbf{j} = 0$ in the boundary layer, hence the layer will be viscous. The solution of Equations (12) and (13) on account of (23) is

$$E_y^{\circ} = \frac{H_0}{c} u, \quad E_x^{\circ} = 0 \tag{25}$$

Hence it follows that the electromagnetic force in (15) is zero

$$f_x = \frac{\sigma}{c} H_0\left(E_y^{\circ} - \frac{H_0}{c} u\right) = 0, \qquad f_y = \frac{\sigma}{c} H_0 E_x^{\circ} = 0$$

i.e. the solution of system (15) likewise describes the usual viscous boundary layer. This example shows the relation between the solutions of system (15) with the solutions of the boundary layer equation and the exact solutions of (1) and (2). The solutions of systems (12) to (13) are sufficient for the calculation of the velocity field in the boundary layer. Equations (12) to (13) are simpler than Equations (1) to (2) and their use permits separation of the boundary layer problem from the external problem. Solutions of system (15) do not differ in the principal terms (in terms of δ) for the velocity field from the solutions of the boundary layer equations by the use of the exact equations (1) and (2).

Solutions of Equations (12) to (13) alone are not sufficient for the determination of the current field. Indeed, from (25) there follows

$$j_x = \frac{\sigma}{c} H_0 v, \qquad j_y = 0$$

which does not agree with $\mathbf{j} = 0$, which is given by the exact equations (1) to (2) and by Ohm's law. This is related to the fact that in the determination of $\mathbf{E}^{\circ}$ from (12) to (13) terms of order δ are not taken into account, while they are taken into account in the calculation of the velocity. In order to calculate $\mathbf{E}$ correctly to terms of the order δ, system (22) is to be used. In the case of the example under consideration, because of (23) and (25) the solution of (22) has the form

$$E_y^1 = 0, \quad E_x^1 = \int \frac{\partial E_y^{\circ}}{\partial x}\, dy = \frac{H_0}{c} \int \frac{\partial u}{\partial x}\, dy = -\frac{H_0}{c} \int \frac{\partial v}{\partial y}\, dy = -\frac{H_0}{c}\, v$$

i.e. solution (22) yields a result which agrees with (24).

In conclusion let it be noted that analogous results may be obtained also in the case of various elaborations of the problem pertaining particularly to Ohm's law, if the evaluations of the basic quantities are not changed.

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